Binomial partial Steiner triple systems with complete graphs: structural problems

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Introduction

In the paper we study the structure of hyperplanes of so called binomial partial Steiner triple systems (BSTS's, in short) i.e. of configurations with $\binom{n}{2}$ points and $\binom{n}{3}$ lines, each line of the size 3. Consequently, a BSTS has $n-2$ lines through each of its points.

The notion of a hyperplane is commonly used within widely understood geometry. Roughly speaking, a hyperplane of a (geometrical) space $\mathfrak{M}$ is a maximal proper subspace of $\mathfrak{M}$. A more specialized characterization of a ("geometrical") hyperplane comes from projective geometry: a hyperplane of a (partial linear = semilinear) space $\mathfrak{M}$ is a proper subspace of $\mathfrak{M}$ which crosses every line of $\mathfrak{M}$. Note that these two characterizations are not equivalent in general. In the context of incidence geometry the second characterization is primarily used (cf. [2] or [20]), and also in our paper in investigations on some classes of partial Steiner triple systems (in short: PSTS's) we shall follow this approach. For a PSTS $\mathfrak{M}$ there is a natural structure of a projective space with all the lines of size 3 definable on the family of all hyperplanes of $\mathfrak{M}$ (the so called Veldkamp space of $\mathfrak{M}$). On other side our previous investigations on PSTS's and graphs contained in them lead us to characterizations of systems which freely contain complete graphs (one can say, informally and not really exactly: systems freely generated by a complete graph); these all fall into the class of so called binomial configurations i.e. $\left(\binom{\nu+\varkappa-1}{\nu}_{\nu}\ \binom{\nu+\varkappa-1}{\varkappa}_{\varkappa}\right)$-configurations with integers $\nu,\varkappa \geq 2$. A characterization of PSTS's which freely contain at least given number m of complete subgraphs appeared available, and for particular values of m a complete classification of the resulting configurations was proved (see [10]). It turned out so, that the structure of complete subgraphs of $\mathfrak{M}$ says much about $\mathfrak{M}$, but fairly it does not determine $\mathfrak{M}$.

Now, quite surprisingly, we have observed that the complement of such a free complete subgraph of a PSTS $\mathfrak{M}$ is a hyperplane of $\mathfrak{M}$. So, our previous classification is equivalent to characterizations and classifications of binomial PSTS's based on the structure of their binomial hyperplanes. But a PSTS, if contains a binomial hyperplane, usually contains also other (non-binomial) hyperplanes. So, the structure of all the hyperplanes of a PSTS $\mathfrak{M}$ says much more about the structure of $\mathfrak{M}$. In the paper we have determined the structure of hyperplanes of PSTS's of some important classes, in particular of so called generalized Desargues configurations (cf.

[3], [4], [17], [21]), of combinatorial Veronese structures and of dual combinatorial Veronese structures, both with 3-element lines (cf. [14], [5]), and of so called combinatorial quasi Grassmannians (cf. [19]). Exact definitions of respective classes of configurations are quoted in the text. Beautiful figures illustrating the schemes of hyperplanes in small PSTS's were prepared by Krzysztof Petelczyc. We have also shown a general method to characterize all the hyperplanes in an arbitrary BSTS with at least one ((maximal) free complete subgraph (Theorems 3.5, 3.9).

As it was said: the hyperplanes of a PSTS yield a projective space $\mathfrak{P}$. In essence, $\mathfrak{P} = PG(n,2)$ for some integer n, so only $n = \dim(\mathfrak{P})$ is an important parameter, but non-isomorphic PSTS's may have the same number $2^{n+1}-1$ of hyperplanes. Consequently, the projective space of hyperplanes of a binomial PSTS $\mathfrak{M}$ does not give a complete information on the geometry of $\mathfrak{M}$.

However, if the points of the $PG(n,2)$, associated with a BSTS, are labelled by the type of geometry that respective hyperplanes carry, the number of nonisomorphic realizations of such labelled spaces drastically decreases. It is pretty well seen in the case of 10_3-configurations, but one can observe it for all BSTS with arbitrary rank of points.

1 Binomial subspaces of a BSTS

A *partial Steiner triple system* (a PSTS) is a partial linear space $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ with the constant point rank and all the lines of the size 3. A *binomial partial Steiner triple system* (a BSTS) is a configuration of the type $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$ for an integer $n \geq 4$; for short, we write B_n for a configuration with these parameters.

The symbols $\wp(X)$ and $\wp_k(X)$ stand for the subsets and the k-subsets of a set X, resp.

1.1 The structure of maximal free subgraphs

A PSTS $\mathfrak{M}$ *freely contains the complete graph* K_X, $X \subset S$ iff for any disjoint 2-subsets $\{a_1, a_2\}$ and $\{b_1, b_2\}$ of X we have $\overline{a_1, a_2} \cap \overline{b_1, b_2} = \emptyset$ ($\overline{a}$ denotes the line of $\mathfrak{M}$ which contains a) and no 3-subset of X is on a line of $\mathfrak{M}$.

Let us recall after [13] some basic properties of BSTS's.

Proposition 1.1. *Let $n \geq 2$ be an integer. A smallest* PSTS *that freely contains the complete graph K_n is a B_{n+1}-configuration. Consequently, it is a* BSTS.

Proposition 1.2. *Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a minimal* PSTS *which freely contains a complete graph $K_X = \langle X, \wp_2(X)\rangle$ and $|X| = n$. Then* the complement of K_X, *i.e. the structure*

$$\mathfrak{M} \setminus X := \langle S \setminus X, \mathcal{L} \setminus \{\overline{e}\colon e \in \wp_2(X)\}\rangle \tag{1}$$

is a B_n-configuration and a subspace of $\mathfrak{M}$.

Conversely, let $\mathfrak{M}$ contain as a subspace a B_n-configuration $\mathfrak{N} = \langle Z, \mathcal{G}\rangle$, Then $S \setminus Z$ yields in $\mathfrak{M}$ a complete K_n-graph freely contained in $\mathfrak{M}$, whose complement is $\mathfrak{N}$.

Proposition 1.3. *Any two distinct complete K_n-graphs freely contained in a B_{n+1}-configuration share exactly one vertex.*

PROPOSITION 1.4. *Let $\langle X_i, \wp_2(X_i)\rangle$, $i = 1,2,3$ be three distinct K_n graphs freely contained in a B_{n+1}-configuration $\mathfrak{M}$. Let $c_k \in X_i \cap X_j$ for all $\{k,i,j\} = \{1,2,3\}$. Then $\{c_1, c_2, c_3\}$ is a line of $\mathfrak{M}$.*

1.2 Algebra of hyperplanes

Let $\mathcal{Z}_1, \mathcal{Z}_2$ be two subsets of a set S. We write (cf. [16])

$$\begin{aligned} \mathcal{Z}_1 \pitchfork \mathcal{Z}_2 &:= (\mathcal{Z}_1 \cap \mathcal{Z}_2) \cup ((S \setminus \mathcal{Z}_1) \cap (S \setminus \mathcal{Z}_2)) && (2) \\ &= S \setminus (\mathcal{Z}_1 \div \mathcal{Z}_2), && (3) \end{aligned}$$

where $\div$ denotes the operation of symmetric difference. Note that identifying a subset $\mathcal{Y}$ of S with its characteristic function $\chi_{\mathcal{Y}}$, and, consequently, identifying S with the constant function $\mathbf{1}$ we can compute simply $S \setminus \mathcal{Y} = \mathbf{1} + \mathcal{Y}$. After that we have $\mathcal{Y}_1 \pitchfork \mathcal{Y}_2 = \mathbf{1} + (\mathcal{Y}_1 + \mathcal{Y}_2)$. Simple computations in the Z_2-algebra of characteristic functions of subsets of S yield immediately the following equations valid for arbitrary subsets $\mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2$ of S:

$$\begin{aligned} \mathcal{Y} \pitchfork \mathcal{Y} &= S, && (4) \\ \mathcal{Y} \pitchfork S &= \mathcal{Y}, && (5) \\ \mathcal{Y}_1 \pitchfork \mathcal{Y}_2 &= \mathcal{Y}_2 \pitchfork \mathcal{Y}_1, && (6) \\ \mathcal{Y}_1 \pitchfork (\mathcal{Y}_1 \pitchfork \mathcal{Y}_2) &= \mathcal{Y}_2, && (7) \\ (\mathcal{Y}_1 \pitchfork \mathcal{Y}_2) \cap \mathcal{Y}_2 &= \mathcal{Y}_1 \cap \mathcal{Y}_2, && (8) \\ (\mathcal{Y} \pitchfork \mathcal{Y}_1) \pitchfork (\mathcal{Y} \pitchfork \mathcal{Y}_2) &= \mathcal{Y}_1 \pitchfork \mathcal{Y}_2, && (9) \end{aligned}$$

Formally, the operation $\pitchfork$ depends on the superset S which contains the arguments of $\pitchfork$. In what follows we shall frequently use this operation without fixing S explicitly: the role of S will be seen from the context.

A *hyperplane of a* PSTS $\mathfrak{M}$ is an arbitrary proper subspace of $\mathfrak{M}$ which crosses every line of $\mathfrak{M}$.

PROPOSITION 1.5. *If H_1, H_2 are distinct hyperplanes of $\mathfrak{M}$ then $H_1 \pitchfork H_2$ is a hyperplane of $\mathfrak{M}$ as well.*

PROOF. Let L be a line of $\mathfrak{M}$. Set $H = H_1 \pitchfork H_2$. Write $L = \{q_1, q_2, q_3\}$. It is seen that, up to a numbering of variables, one of the following must occur:

(i) $q_1 \in H_1, H_2$, $q_2, q_3 \notin H_1 \cup H_2$: then $L \subset H$.

(ii) $q_1 \in H_1, H_2$, $q_2, q_3 \in H_1 \setminus H_2$.

(iii) $q_1, q_2, q_3 \in H_1, H_2$: clearly, $L \subset H$.

(iv) $q_1 \in H_1 \setminus H_2$, $q_2 \in H_2 \setminus H_1$, $q_3 \notin H_1, H_2$: then $q_3 \in H$.

In each case L crosses H, and if L has two points in H then L is contained in H. □

Let $\mathcal{H}(\mathfrak{M})$ be the set of hyperplanes of $\mathfrak{M}$. Note, in addition, that the structure

$$\boldsymbol{V}(\mathfrak{M}) := \langle \mathcal{H}(\mathfrak{M}), \{\{H_1, H_2, H_1 \pitchfork H_2\} \colon H_1, H_2 \in \mathcal{H}(\mathfrak{M}),\ H_1 \neq H_2\}\rangle \qquad (10)$$

is a projective space $PG(n, 2)$, possibly degenerated i.e. with $n = -1, 0, 1$ allowed. This projective space will be referred to as *the Veldkamp space of* $\mathfrak{M}$ (cf. [2], [22], (or [20])).

As a by-product we get that *for each* PSTS $\mathfrak{M}$, $|\mathcal{H}(\mathfrak{M})| = 2^{n+1} - 1$ *for an integer* $n \geq -1$.

For an arbitrary set X and $\emptyset \neq A', A'' \subset X$ we write

$$\mathcal{H}(A'|A'') := \wp_2(A') \cup \wp_2(A''). \tag{11}$$

The following set-theoretical formula

$$\begin{aligned}\mathcal{H}(A|X \setminus A) \pitchfork \mathcal{H}(B|X \setminus B) = \\ \mathcal{H}\Big((A \cap B) \cup ((X \setminus A) \cap (X \setminus B))\Big|(A \cap (X \setminus B)) \cup (B \cap (X \setminus A))\Big) \\ = \mathcal{H}(A \div B|X \setminus (A \div B)) \end{aligned} \tag{12}$$

is valid for any distinct $\emptyset \neq A, B \subsetneq X$.

Let us fix $Z \subsetneq X$, X – finite. The following is just a simple though important observation.

***Remark* 1.** The set

$$\{\mathcal{H}(\{i\}|X \setminus \{i\}) \colon i \in Z\}$$

generates via $\pitchfork$ the subalgebra

$$D(Z) = \{\mathcal{H}(A|X \setminus A) \colon \emptyset \neq A \subset Z\}$$

of the $\pitchfork$-algebra

$$D(X) = \{\mathcal{H}(A|X \setminus A) \colon \emptyset \neq A \subsetneq X\}.$$

Clearly, $D(Z)$ determines as in (10) a Fano projective space $PG(n, 2)$, a subspace of the projective space in the analogous way associated with $D(X)$. Both algebras and both projective spaces up to an isomorphism depend entirely on the cardinalities $|Z|$ and $|X|$.

1.3 Binomial subconfigurations

Next, we continue investigations of Subsection 1.1, but now we concentrate upon the 'complementary configurations' contained in a BSTS; in view of 1.2, this is an equivalent approach.

Let us begin with a few words on basic properties of such a complementary configuration.

PROPOSITION 1.6. *Let X be a K_n-graph freely contained in a B_{n+1}-configuration $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ and let $Y = S \setminus X$ be the corresponding B_n-subconfiguration, complementary to K_X. Then Y is a hyperplane of $\mathfrak{M}$.*

PROOF. Let $L \in \mathcal{L}$, if $x \in L \cap Y$ or $x, y \in L \cap Y$ are given, then $L \cap Y \neq \emptyset$: by assumptions. Suppose there are two $x, y \in L$, $x, y \notin Y$. So, $x, y \in X$, by assumptions on X we have $\overline{x, y} \setminus X \in L \cap Y$. □

Roughly speaking, establishing the structure which maximal complete graphs yield in a B_{n+1}-BSTS consists in establishing the structure which maximal binomial subspaces yield in the configuration, which can be equivalently reformulated as

establishing the structure of binomial hyperplanes in the binomial configuration. So, the subject of this paper is the problem known as *hyperplanes arrangements* in binomial partial Steiner triple systems. The question if each hyperplane is a complete-graph-complement has, generally a negative solution: indeed (cf. 4.1), if $|A|, |X \setminus A| > 2$ then $\mathcal{H}(A|X \setminus A)$ defined by (11) is not a binomial configuration, though it happens to be even a hyperplane. A first counterexample is given in 4.1. A more general argument follows by 1.5 and the following observation.

PROPOSITION 1.7. *Let Y_1, Y_2 be complements of distinct maximal complete graphs X_1, X_2 freely contained in a* BSTS $\mathfrak{M} = \langle S, \mathcal{L} \rangle$, $|S| > 3$. *Then $Y_1 \pitchfork Y_2$ is a hyperplane of $\mathfrak{M}$ which is not the complement of any complete subgraph of $\mathfrak{M}$.*

PROOF. Set $Y = Y_1 \pitchfork Y_2$; it suffices to note that the complement $X = S \setminus Y$ of Y is not a complete graph. It is seen that

$$X = (X_1 \cap Y_2) \cup (X_2 \cap Y_1).$$

If $x, y \in X_i$, $x \neq y$ then, by definition, there is $z \in Y_i$ such that $\{x, y, x\}$ is a line of $\mathfrak{M}$; we write $z = \{x, y\}^{\infty_i}$. Take any distinct $x, y \in S$.
If $x, y \in X_1, Y_2$ then x, y are joinable; let $z = x \oplus y$ be the third element of $\overline{x, y}$. Then $z \in \overline{x, y} \subset Y_2$ and $z = \{x, y\}^{\infty_1} \in Y_1$.
Analogously, if $x, y \in X_2, Y_1$ then the third point $z = x \oplus y$ of $\overline{x, y}$ lies on $Y_1 \cap Y_2$.
Take $x \in X_1, Y_2$, $y \in X_2, Y_1$ and suppose there is a line through x, y; again we take $z = x \oplus y$. By the above, $z \notin X_1 \cap Y_2, X_2 \cap Y_1, Y_1 \cap Y_2$. So, only the case $z \in X_1 \cap X_2$ remains to be examined.
By 1.3, $X_1 \cap X_2 = \{c\}$ for a point c. So, finally, we take $x \in X_1 \cap Y_2$ collinear with c, and $y \in X_2 \cap Y_1$, $y \notin \overline{x, c}$ and then $x, y \in X$ are not collinear in $\mathfrak{M}$. □

2 Examples: hyperplanes in 10_3-configurations

To give intuitions how the hyperplanes in well known configurations look like we enclose this Section. The following can be proved by a direct inspection of all the 10_3 configurations. Some of these facts follow from more general theory developed in next sections, but we give them right at the beginning to give intuitions how the theory looks like. It is known that there are exactly ten 10_3-configurations (see e.g. [12], [1], [11], [13]); names of the configurations in question are used mainly after [8]).

PROPOSITION 2.1. *(schemes of Veldkamp spaces of the configurations enumerated below are presented in figures 1-6)*

(i) *Desargues configuration has fifteen hyperplanes (cf. [21]).*

(ii) *The Kantor 10_3G-configuration (Fig. 1) and the nightcap configuration (Fig. 6) have seven hyperplanes each.*

(iii) *The fez configuration (Fig. 4) and the headdress configuration (Fig. 5) contain three hyperplanes each.*

(iv) *The basinet configuration (Fig. 2) and the overseashat configuration (Fig. 3) contain exactly one hyperplane each.*

(v) *Every of the remaining three 10_3 configurations does not contain any hyperplane.*

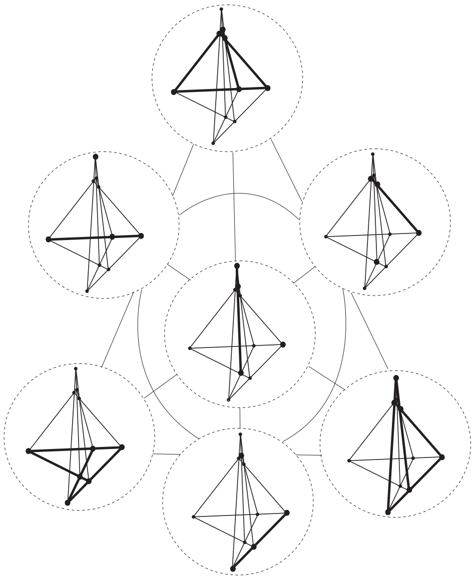

Figure 1: The Veldkamp Space of the Kantor 10_3G-Configuration $\mathbf{V}_3(3)$

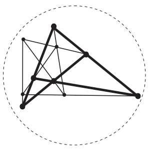

Figure 2: The Veldkamp space of the basinet configuration

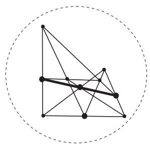

Figure 3: The Veldkamp space of the overseas-cap configuration

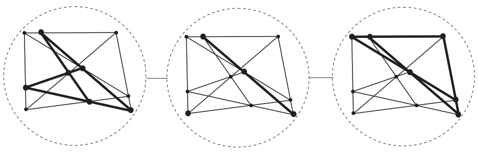

Figure 4: The Veldkamp space of the fez configuration

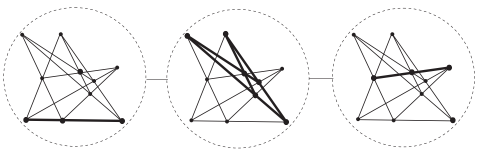

Figure 5: The Veldkamp space of the headdress configuration

As a consequence we can formulate

***Remark* 2.** There are binomial configurations (even quite small: 10_3-configurations) with exactly one hyperplane. This one may be the complement of a complete

graph (a 10_3-configuration with exactly one Veblen subconfiguration) or not (a 10_3-configuration whose unique hyperplane consists of a point and a line).

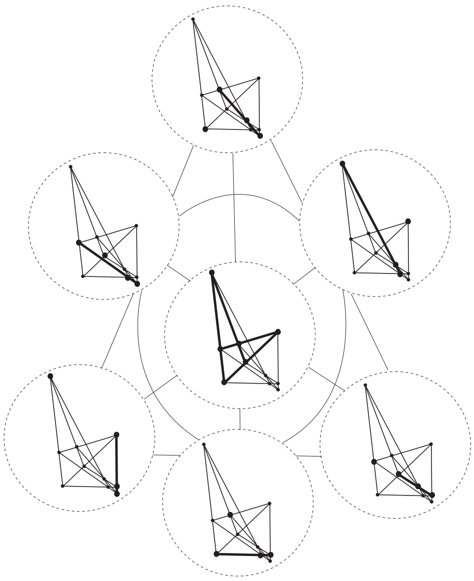

Figure 6: The Veldkamp space of the nightcap configuration

3 A general approach: Hyperplanes of binomial PSTS's with a maximal complete graph inside

Let us begin with the formal construction which makes more precise the statements of 1.2. Let $|X| = n$ and $0 \notin X$, $W = X \cup \{0\}$. Every BSTS $\mathfrak{M}$ with $\binom{n+1}{2}$ vertices and K_X freely contained in it can be presented in the form $K_X +^{\mu} \mathfrak{V}$, defined below:

let $\mathfrak{V}$ be a B_n-configuration and μ be a bijection of $\wp_2(X)$ onto the point set of $\mathfrak{V}$. The point set of $K_X +^{\mu} \mathfrak{V}$ is the union of the set of vertices of K_X and the point set of $\mathfrak{V}$. The set of lines of $K_X +^{\mu} \mathfrak{V}$ is the union of the set of lines of $\mathfrak{V}$ and the family $\{\{x, y, \mu(\{x, y\})\} \colon x, y \in X, x \neq y\}$. Up to an isomorphism, $K_X +^{\mu} \mathfrak{V}$ can be in a natural way defined on the set $\wp_2(W)$ as its point set: we identify each $x \in X$ with the set $\{0, x\}$, and identify each point $\mu(\{x, y\})$ of $\mathfrak{V}$ with the set $\{x, y\}$, suitably transforming the line set of $\mathfrak{V}$ and putting, formally, $\mu(\{x, y\}) = \{x, y\}$. Frequently, we write $(x, y)^{\infty} = \overline{x, y}^{\infty} = \mu(\{x, y\})$ for distinct $x, y \in X$.

In the first step we shall characterize hyperplanes in a configuration $\mathfrak{M} = K_X +^{\mu} \mathfrak{V}$. So, let H be a hyperplane of $\mathfrak{M}$. Let V be the point set of $\mathfrak{V}$; then $H_0 = H \cap V$ is a hyperplane of $\mathfrak{V}$ or $H_0 = V$. We begin with several technical lemmas. For $x, y \in X$ we write $x \sim y$ when $x \neq y$ and $(x, y)^{\infty} \in H_0$ or $x = y \in X$.

Lemma 3.1. *The relation $\sim$ is an equivalence relation.*

Proof. It is evident that $\sim$ is symmetric and reflexive. So it remains to prove the transitivity of $\sim$. Let $x, y, z \in X$ be pairwise distinct. Assume that $(x, y)^{\infty}, (x, z)^{\infty} \in H_0$ and suppose that $(y, z)^{\infty} \notin H_0$. Then $H \cap \overline{y, z} \subset \{y, z\}$, as H crosses *every* line

of $\mathfrak{M}$. Assume $y \in H$; from $\overline{y, (x,y)^\infty} \subset H$ we infer $x \in H$ and then $z \in H$ follows. Finally, $\overline{y,z} \subset H$, so $(y,z)^\infty \in H_0$. □

LEMMA 3.2. *Let $x \in X$. If there is $z \in [x]_\sim \cap H_0$ then $[x]_\sim \subset H$.*

PROOF. From assumptions, $[x]_\sim = [z]_\sim$. Let $y \sim z$ be arbitrary. Then $(y,z)^\infty \in H$ and $z \in H$ yield $y \in H$. □

Write $\mathcal{X} = X/\!\sim$. From 3.2 we know that

$$\text{for every } \mathfrak{a} \in \mathcal{X}, \text{ either } \mathfrak{a} \subset H \text{ or } \mathfrak{a} \cap H = \emptyset.$$

LEMMA 3.3. *For every $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ if $\mathfrak{a}, \mathfrak{b} \subset H$ then $\mathfrak{a} = \mathfrak{b}$.*

PROOF. Let $x, y \in X$ such that $\mathfrak{a} = [x]_\sim$ and $\mathfrak{b} = [y]_\sim$. Let $x \neq y$. From assumptions, $x, y \in H$ and then $\overline{x,y} \subset H$ gives $(x,y)^\infty \in H$ i.e. $x \sim y$, as required. □

LEMMA 3.4. *For every distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$, $\mathfrak{a} \subset H$ or $\mathfrak{b} \subset H$.*

PROOF. Let $x, y \in X$ such that $\mathfrak{a} = [x]_\sim$ and $\mathfrak{b} = [y]_\sim$. From assumptions, $x \not\sim y$ i.e. $(x,y)^\infty \notin H$. But $\overline{x,y}$ crosses H, so $x \in H$ or $y \in H$. From 3.2 we get the claim. □

Now, we are in a position to prove the first (main) characterization.

THEOREM 3.5. *Let H be a hyperplane of $\mathfrak{M} = K_X +^\mu \mathfrak{V}$ defined on the set $\wp_2(W)$, as introduced at the beginning of the section. Then there is a subset A of W such that $H = \mathcal{H}(A|W \setminus A)$.*

PROOF. From 3.3 and 3.4 we get that either $\sim$ has exactly two equivalence classes $\mathfrak{a} \subset H, \mathfrak{b} \subset X \setminus H$ or $\mathcal{X} = \{X\}$. In the second case, $V \subset H$, and if there were $x \in H \cap X$ then H is the point set of $\mathfrak{M}$. So, $H = V = \wp_2(X) = \mathcal{H}(\{0\}|W \setminus \{0\})$. Let us pass to the first case. Note that H is the union of three sets: $\mu(\wp_2(\mathfrak{a})) = \wp_2(\mathfrak{a})$, $\mu(\wp_2(\mathfrak{b})) = \wp_2(\mathfrak{b})$, and $\mathfrak{a}$ which, under identification introduced before, corresponds to $\{\{0,x\} \colon x \in \mathfrak{a}\}$. So, finally, H can be written in the form $\wp_2(\mathfrak{b}) \cup \wp_2(\mathfrak{a} \cup \{0\}) = \mathcal{H}(\mathfrak{b}|W \setminus \mathfrak{b})$. □

Next, we are going to determine which "bipartite" sets $\mathcal{H}(A|X \setminus A)$ are hyperplanes of suitable BSTS's. To this aim one should know more precisely what is the number of complete graphs inside a given configuration.

Let us recall the following construction

Let $I = \{1, \ldots, m\}$ be arbitrary, let $n > m$ be an integer, and let X be a set with $n - m + 1$ elements. Let us fix an arbitrary B_{n-m+1}-configuration $\mathfrak{B} = \langle Z, \mathcal{G} \rangle$. Assume that we have two maps μ, ξ defined: $\mu \colon I \longrightarrow Z^{\wp_2(X)}$ and $\xi \colon I \times I \longrightarrow S_X$, such that $\xi_{i,i} = \mathrm{id}$, $\xi_{i,j} = \xi_{j,i}^{-1}$, and μ_i is a bijection for all $i, j \in I$. Set $S = Z \cup (X \times I) \cup \wp_2(I)$ (to avoid silly errors we assume that the given three sets are pairwise disjoint). On S we define the following family $\mathcal{L}$ of blocks

$$\begin{aligned} \mathcal{L} &= \mathcal{G} && (13)\\ &\cup \ \text{the lines of } \mathbf{G}_2(I) && (14)\\ &\cup \ \{\{\{i,j\}, (x,i), (\xi_{i,j}(x), j)\} \colon \{i,j\} \in \wp_2(I), x \in X\} && (15)\\ &\cup \ \{\{(a,i), (b,i), \mu_i(\{a,b\})\} \colon \{a,b\} \in \wp_2(X), i \in I\}. && (16) \end{aligned}$$

Write $m \bowtie_{\xi}^{\mu} \mathfrak{B} = \langle S, \mathcal{L} \rangle$. It needs only a straightforward (though quite tedious) verification to prove that $\mathfrak{M} := m \bowtie_{\xi}^{\mu} \mathfrak{B}$ *is a* B_{n+1}*-configuration.*

For each $i \in I$ we set $Z_i = X \times \{i\}$, $S_i = \{e \in \wp_2(I) \colon i \in e\}$, and $X_i = Z_i \cup S_i$. Then $\mathfrak{M}$ *freely contains* m K_n*-graphs; these are* $X_1, \ldots, X_m$. It is seen that the point $\{i, j\}$ is the "perspective center" of two subgraphs Z_i, Z_j of $\mathfrak{M}$. So, we call the arising configuration *a system of perspectives of* $(n - m + 1)$*-simplices.* Define $\mu_i \colon \wp_2(Z_i) \longrightarrow Z$ by the formula $\mu_i(\{(x, i), (y, i)\}) = \mu(i)(\{x, y\})$; the configuration $\mathfrak{B}$ is the common 'axis' of the configurations $\langle Z_i, \wp_2(Z_i) \rangle +_{\mu_i} \mathfrak{B}$ contained in $\mathfrak{M}$. Let us denote $W = X \cup I$. Without loss of generality we can assume that $Z = \wp_2(X)$ and each $(x, i) \in X \times I$ can be identified with the set $\{x, i\}$. After this identification $\wp_2(W)$ becomes the point set of $\mathfrak{M}$.

The following is crucial:

Theorem 3.6 ([13])**.** *Let* $\mathfrak{M}$ *be a* B_{n+1}*-configuration.* $\mathfrak{M}$ *freely contains (at least)* m K_n*-graphs iff* $\mathfrak{M} \cong m \bowtie_{\xi}^{\mu} \mathfrak{B}$ *for a* B_{n-m+1}*-configuration* $\mathfrak{B}$ *and a pair* (μ, ξ) *of suitable maps.*

Combining the results of [13] and [10] it is not too hard to prove the following criterion

Proposition 3.7. *Let* $\mathfrak{M} = m \bowtie_{\xi}^{\mu} \mathfrak{B}$ *for a* B_{n-m+1}*-configuration* $\mathfrak{B}$ *and a pair* (μ, ξ) *of suitable maps. The following conditions are equivalent*

(i) $\mathfrak{M}$ *freely contains at least* $m + 1$ K_n *graphs.*

(ii) $\mathfrak{M}$ *contains at least* $m + 1$ B_n*-subconfigurations.*

(iii) *There is* $x_0 \in X$ *such that* $\xi(i, j)(x_0) = x_0$ *for each pair* $i, j \in I$*.*

From now on we assume that

$\mathfrak{B}$ has $\wp_2(X)$ as its point set and $\mathfrak{M} = m \bowtie_{\xi}^{\mu} \mathfrak{B}$, defined on the point set $\wp_2(W)$, freely contains exactly m K_n-subgraphs;

this means that $\mathcal{H}(i) = \mathcal{H}(\{i\}|W \setminus \{i\})$ with $i \in I$ are the unique hyperplanes of $\mathfrak{M}$ of the size $\binom{n}{2}$.

From the above, 1.5, and 1 we infer immediately

Proposition 3.8. (i) *Every set* $\mathcal{H}(J|W \setminus A)$ *with* $J \subset I$ *is a hyperplane of* $\mathfrak{M}$*. In particular,* $\mathcal{H}(I|X)$ *is a hyperplane of* $\mathfrak{M}$*.*

(ii) *There is no* $a \in X$ *such that* $\mathcal{H}(\{a\}|W \setminus \{\, a\})$ *is a hyperplane of* $\mathfrak{M}$*.*

With the help of (12) we get

$$\mathcal{H}(A \cup J|W \setminus (A \cup J)) = \mathcal{H}(A|W \setminus A) \pitchfork \mathcal{H}(J|W \setminus J)$$

for every $J \subset I$, $A \subset X$. So, from 3.8 we get

if $J \subset I$, $A \subset X$ then
$\mathcal{H}(A \cup J|W \setminus (A \cup J))$ is a hyperplane of $\mathfrak{M}$ iff $\mathcal{H}(A|W \setminus A)$ is a hyperplane of $\mathfrak{M}$.

Theorem 3.9. *Let* $A \subset X$*. Then* $\mathcal{H}(A|W \setminus A)$ *is a hyperplane of* $\mathfrak{M}$ *iff the following conditions are satisfied:*

(i) $\wp_2(A)$ *is a hyperplane of* $\mathfrak{B}$*.*

(ii) *A is invariant under every $\xi(i,j)$, $i,j \in I$.*

(iii) *A is invariant under every μ_i, $i \in I$, which means the following: if $\mu_i(x,y) = \{u,v\}$ then $\{x,y\} \subset A$ or $\{x,y\} \subset X \setminus A$ iff $\{u,v\} \subset A$ or $\{u,v\} \subset X \setminus A$.*

Note 1. *In some applications there is no way to present, in a natural way, the underlying B_{n-m+1}-configuration $\mathfrak{B}$ as a structure defined on the family of 2-subsets of a $(n-m+1)$-element set;* natural *from the point of view of the geometry of $\mathfrak{B}$. Then one can take one of μ_i's as basic and replace $\mathfrak{B}$ as its coimage under μ_i defined on $\wp_2(X)$. Under this stipulation the condition* (iii) *of 3.9 is read as follows:*

(iii') if $\mu_i(x,y) = \mu_j(u,v)$ for some $i,j \in I$ then $\{x,y\} \subset A$ or $\{x,y\} \subset X \setminus A$ iff $\{u,v\} \subset A$ or $\{u,v\} \subset X \setminus A$.

Proof. We use notation of the definition of a system of perspectives of simplices presented in this Section. The symbol $a \oplus b$ means the third point $\overline{a,b} \setminus \{a,b\}$ on the line through a,b (if the line exists). Note that $\wp_2(W \setminus A) = \wp_2(I) \cup \wp_2(X \setminus A) \cup (X \setminus A) \boxtimes I$, where $(X \setminus A) \boxtimes I := \{\{b,i\} \colon i \in I,\, b \in X \setminus A\}$. Therefore,

$$H = \mathcal{H}(A|W \setminus A) = \wp_2(A) \cup \wp_2(I) \cup \wp_2(X \setminus A) \cup (X \setminus A) \boxtimes I,$$

where $A \subset X$. In view of 3.8(i) without loss of generality we can assume that $A \neq X$.

Since $\wp_2(I) \subset H$,

$$\text{if a line } L \text{ of } \mathfrak{M} \text{ has two points common with } \wp_2(I) \text{ then } L \subset H, \\ \text{and each line of } \mathfrak{M} \text{ of the form (14) crosses } H. \quad (17)$$

Assume that H is a hyperplane of $\mathfrak{M}$. Since $\wp_2(X)$ is the point set of $\mathfrak{B}$, and $\mathfrak{B}$ is a subspace of $\mathfrak{M}$ right from definition, $\mathcal{H}(A|X \setminus A) = \wp_2(A) \cup \wp_2(X \setminus A)$ is a hyperplane of $\mathfrak{B}$ or $\wp_2(X) = \wp_2(A)$. The latter means $A = X$, which contradicts assumptions. So, (i) follows.

On the other hand, converting the above reasoning we easily prove that (i) implies

$$\text{if a line } L \text{ of } \mathfrak{M} \text{ has two points common with } \wp_2(A) \cup \wp_2(X \setminus A) \text{ then } L \subset H, \\ \text{and each line of } \mathfrak{M} \text{ of the form (13) crosses } H. \quad (18)$$

Next, let us pass to the lines of $\mathfrak{M}$ of the form (15). Suppose that $a \notin A$, $i_1, i_2 \in I$, $i_1 \neq i_2$. Then $\{a,i_1\}, \{i_1,i_2\} \in H$, so $\{a,i_1\} \oplus \{i_1,i_2\} = \{\xi(i_1,i_2)(a), i_2\} \in H$. Consequently, $\xi(i_1,i_2)(a) \notin A$. This justifies condition (ii).

Considering all the points expressible in the form $\{a,i_1\}, \{i_1,i_2\}, \{a',i_2\} \in H$, i.e. with $a, a' \notin A$, $i_1, i_2 \in I$, $i_1 \neq i_2$ and afterwards considering their 'product' $\{a,i_1\} \oplus \{i_1,i_2\}$ and $\{a,i_1\} \oplus \{a',i_2\}$ we see that, conversely, (iii) implies

$$\text{if a line } L \text{ of } \mathfrak{M} \text{ has two points common with } H, \text{ two in } (X \setminus A) \boxtimes I \\ \text{or one in } (X \setminus A) \boxtimes I \text{ and the second in } \wp_2(I) \text{ then } L \subset H, \\ \text{and each line of } \mathfrak{M} \text{ of the form (15) crosses } H. \quad (19)$$

Finally, we pass to the lines of $\mathfrak{M}$ of the form (16). Let $p = \{a_1, a_2\} \in \wp_2(X)$, and $b \in X$, $i \in I$, $q = \{i, b\}$. Suppose p, q are collinear in $\mathfrak{M}$; this means $\mu_i(b, b') =$

$\{a_1, a_2\}$ for a point $b' \in X$ and $r := p \oplus q = \{i, b'\}$. Set $d = \{b, b'\}$. Assume that $p \in H$ i.e. $p \subset A$ or $p \subset X \setminus A$. Then $q \in H$ iff $r \in H$ i.e. $q \in (X \setminus A) \boxtimes I$ iff $r \in (X \setminus A) \boxtimes I$. Finally: $b \in A$ iff $b' \in A$, so $d \subset A$ or $d \subset (X \setminus A)$ follows.

Next, let $q \in H$. Then $p \in H$ iff $r \in H$ yields that the implication ($b, b' \in A$ or $b, b' \notin A \implies p \subset A$ or $p \subset X \setminus A$) holds. So, finally, we have proved (iii).

Converting the reasonings above we see that (iii) implies

$$\begin{gathered}\text{if a line } L \text{ of } \mathfrak{M} \text{ has two points common with } H, \text{ two in } (X \setminus A) \boxtimes I \\ \text{or one in } (X \setminus A) \boxtimes I \text{ and the second in } \wp_2(A) \cup \wp_2(X \setminus A) \text{ then } L \subset H, \\ \text{and each line of } \mathfrak{M} \text{ of the form (16) crosses } H.\end{gathered} \tag{20}$$

Gathering together the conditions (17), (18), (19), and (20) we obtain that the conjunction (i) & (ii) & (iii) implies that H is a hyperplane of $\mathfrak{M}$. □

There do exist PSTS's which satisfy the assumptions (i)-(iii); as examples known in the literature we can quote quasi Grassmannians, comp. Subsect. 4.2 and, in particular, 4.4. Another class of examples is shown in 3.10.

EXAMPLE 3.10. Let $I = \{1, \ldots, m\}$, $X = \{a, a, b, b'\}$ and $\mathfrak{B} = \mathbf{G}_2(X)$ be the Veblen configuration. Set $\xi(i,j) = \xi(j,i)(a, a', b, b') = (a', a, b', b)$, $\mu_i(x, y) = \{x, y\}$ for all $i, j \in I$, $x, y \in X$. Let us put $\mathfrak{M} := m \bowtie_\xi^\mu \mathfrak{B}$. Then $\mathfrak{M}$ is a system of perspectives of m tetrahedrons. It freely contains exactly m graphs K_{m+3}, so it contains exactly m hyperplanes of the form $\mathcal{H}(\{x\}|X \setminus \{x\}) = \mathcal{H}(x)$ with $x \in I \cup X$. However, it contains the hyperplane $\mathcal{H}(\{a, a'\}|\{b, b'\} \cup I)$ which is not $\pitchfork$-generated from the $\mathcal{H}(x)$'s.

We close this section with a characterization of geometries on hyperplanes $\mathcal{H}(A \cup J|B \cup E)$ of $\mathfrak{M}$, where $\{A, B\}$ is a decomposition of X and $\{J, E\}$ is a decomposition of I. So, let us assume that (i)-(iii) of 3.9 hold.

PROPOSITION 3.11. *Let $k = |J|$, $m = |I|$.*

(i) *$\mathcal{H}(J|W \setminus J)$ is the union of the generalized Desargues configuration $\mathbf{G}_2(J)$ and the system $k \bowtie_{\xi \restriction J \times J}^{\mu \restriction J} \mathfrak{B}$ of perspectives of k simplices K_X.*

(ii) *$\mathcal{H}(A|W \setminus A)$ is the union of the restriction $\mathfrak{B} \restriction \wp_2(A) =: \mathfrak{B}'$ and the system $m \bowtie_{\xi'}^{\mu'} \mathfrak{B}'$, where $\xi'(i,j) = \xi(i,j) \restriction A$ and $\mu'(i) = \mu(i) \restriction \wp_2(A)$ for all $i, j \in I$, of perspectives of m simplices K_A.*

4 Examples: structure of hyperplanes in BSTS's of some known classes

4.1 Hyperplanes in generalized Desargues configurations

Recall: a B_n-configuration $\mathfrak{M}$ freely contains n graphs K_{n-1} (the maximal possible amount) iff $\mathfrak{M}$ is isomorphic to the *generalized Desargues configuration* $\mathbf{G}_2(X) = \langle \wp_2(X), \{\wp_2(Z) \colon Z \in \wp_3(X)\} \rangle$ for a X with $|X| = n$ (cf. [13], [10]). The class of generalized Desargues configurations appears in many applications, even in physics: [3], [4].

Theorem 4.1. *Let $H \subset \wp_2(X)$. Write $\mathfrak{H} = \mathbf{G}_2(X)$. The following conditions are equivalent*

(i) *H is a hyperplane of $\mathfrak{H}$.*

(ii) *There is a proper non void subset Z of X such that $H = \mathcal{H}(Z|X \setminus Z)$.*

Consequently, $\boldsymbol{V}(\mathbf{G}_2(n)) = PG(n-2, 2)$ (comp. [21]).

Proof. Let H be as required in (ii) and let $L = \wp_2(A)$ for a $A \in \wp_3(X)$ be a line of $\mathfrak{H}$. If $A \subset Z$ or $A \subset X \setminus Z$ then $L \subset H$. Assume that $A \not\subset Z, X \setminus Z$. Then there are $i, j \in A$, $i \in\notin Z$, $j \notin (X \setminus Z)$ So: $i \in X \setminus Z$, $j \in Z$. Write $A = \{i, j, l\}$. If $l \in Z$ then $\{j, l\} \in L \cap H$, if $l \in X \setminus Z$ then $\{i, l\} \in L \cap H$. Finally, we note that if $L \cap \wp_2(Z) \neq \emptyset$ then there is no point in $L \cap \wp_2(X \setminus Z)$. And similarly conversely. This proves that H is a subspace of $\mathfrak{G}$, so, finally, (i) is valid.

The implication (i) $\Longrightarrow$ (ii) is immediate after 3.5.

Finally, there are $\frac{2^n-2}{2} = 2^{n-1}-1$ suitable decompositions of X; this determines $\dim(\boldsymbol{V}(\mathbf{G}_2(X)))$. □

Recall after [17] that each set

$$S(i) = \{a \in \wp_2(X) \colon i \in a\} \tag{21}$$

is a complete graph freely contained in $\mathbf{G}_2(X)$. In consequence of 1 and 4.1, the hyperplanes $\mathcal{H}(i) = \mathcal{H}(\{i\}|X \setminus \{i\})$ with $i \in X$ ('binomial hyperplanes' of $\mathbf{G}_2(X)$) generate via the operation $\pitchfork$ all the hyperplanes of $\mathbf{G}_2(X)$.

Note that each of the two components $\mathcal{A} = \wp_2(A)$ and $\mathcal{A}' = \wp_2(A')$ with $A' = X \setminus A$ of $\mathcal{H}(A|A')$ is a binomial configuration. These two components are *complementary* (unconnected) in the following sense:

if $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$ then a, a' are uncollinear in $\mathbf{G}_2(X)$; $\mathcal{A}'$ consists of the points that are uncollinear with every point in $\mathcal{A}$, and conversely.

4.2 Hyperplanes of quasi Grassmannians

First, we recall after [19] a construction of quasi Grassmannians.

Let us fix two sets: Y such that $|Y| = 2(k-1)$ for an integer k and X_0 such that $X_0 \cap Y = \emptyset$, $X_0 = \{1, 2\}$ or $X_0 = \{0, 1, 2\}$. We put $X = X_0 \cup Y$; Then $n := |X| = 2k$ or $n = 2k+1$, resp. The points of the *quasi Grassmannian* $\mathfrak{R}_n$ are the elements of $\wp_2(X)$. The lines of $\mathfrak{R}_n$ are of two sorts: the lines of $\mathbf{G}_2(X)$ which miss $\{1, 2\} =: p$ remain unchanged. The class of lines of $\mathbf{G}_2(X)$ through p (i.e. the sets $\wp_2(Z)$ with $1, 2 \in Z \in \wp_3(X)$) is removed; instead, we add the following sets $\{\{1, 2\}, \{1, 2j+2\}, \{2, 2j+1\}\}$, $\{\{1, 2\}, \{1, 2j+1\}, \{2, 2j+2\}\}$ (we adopt a numbering of Y so that $Y = \{3, 4, 5, 6, \ldots, 2k\}$). It is seen that $\mathfrak{R}_n$ is a B_n-configuration.

Proposition 4.2 ([19]). *The maximal complete K_{n-1}-graphs contained in $\mathfrak{R}_n$ are exactly all the sets $S(i)$ (as defined by* (21)*) with $i \in X_0$.*

Write $D(i) := \wp_2(X) \setminus S(i) = H(\{i\}|X \setminus \{i\})$. Then the following is immediate

Fact 4.3. *When $n = 2k+1$, then $D(0) \cong \mathfrak{R}_{2k}$. For every n, $D(1)$ and $D(2)$ yield in $\mathfrak{R}_n$ (binomial) subconfigurations isomorphic to $\mathbf{G}_2(n-1)$.*

THEOREM 4.4. *Set $T = \{2, 3, \ldots, k\}$, $q_t = \{2t, 2t-1\}$ for $t \in T$. Then $Y = \bigcup_{t \in T} q_t$. The family $\mathcal{H}(\mathfrak{R}_n)$ consists of the sets*

$$\mathcal{H}\left(A \cup \bigcup_{t \in J} q_t \middle| (X_0 \setminus A) \cup \bigcup_{s \in T \setminus J} q_s\right) \tag{22}$$

with arbitrary $A \subseteq X_0$, $J \subseteq T$ such that $(A, J) \neq (\emptyset, \emptyset), (X_0, T)$. Consequently, $\boldsymbol{V}(\mathfrak{R}_{2k}) = PG(k-1, 2)$ and $\boldsymbol{V}(\mathfrak{R}_{2k+1}) = PG(k, 2)$.

PROOF. An elementary computation shows that each set of the form (22) is a hyperplane of $\mathfrak{R}_n$. Let us represent $\mathfrak{R}_n$ as a suitable system of perspectives. Define, first, for $i \in X_0$ and distinct $x, y \in Y$: $\mu_i(\{x, y\}) = \{x, y\}$. Next, we set $\xi(1,2)(2j+1, 2j+2) = (2j+2, 2j+1)$ for every $j = 1, ..., k-1$ and $\xi(1,0) = \xi(2,0)(x) = x$ for $x \in Y$, if $0 \in X_0$. We have obtained two maps $\mu\colon X_0 \longrightarrow \wp_2(Y)^{\wp_2(Y)}$ and $\xi\colon X_0 \times X_0 \longrightarrow S_Y$. It is seen that $\mathfrak{R}_n \cong m \bowtie_\xi^\mu \mathbf{G}_2(X_0)$. A ξ-invariant subset of Y is the union of several sets of the form q_t, $t \in T$. It is seen that such a union is μ-invariant. From 3.9 we infer that each hyperplane of $\mathfrak{R}_n$ has form (22).

To complete the proof it suffices to note that there are $2^{|X_0|}$ decompositions of X_0, 2^{k-1} decompositions of T, and $\frac{2^{|X|+k-1}}{2} - 1 = 2^{|X_0|+k-2} - 1$ decompositions of W which yield a hyperplane. Substituting $|X_0| = 2$ and $|X_0| = 3$ we get $|\mathcal{H}(\mathfrak{R}_{2k})| = 2^k - 1$ and $|\mathcal{H}(\mathfrak{R}_{2k+1})| = 2^{k+1} - 1$, which closes the proof. □

4.3 Hyperplanes of multi-veblen configurations

Recall another fact: a B_n-configuration freely contains $n-2$ graphs K_{n-1} if it is a *(simple) multi-veblen configuration.* The multi-veblen configurations can be also defined by means of a direct construction. Let us recall, briefly, after [15] this construction.

Let X be an n-set disjoint with a two-element set p and $\mathcal{P}$ be a graph defined on X. The points of the configuration $\mathbf{M}(X, p, \mathcal{P})$ are the following: p, s_i with $s \in p$, $i \in X$, and $c_{i,j}$ with $\{i, j\} \in \wp_2(X)$. The lines are: the sets of the form $\{p, a_i, b_i\}$, the sets $\{a_i, a_j, c_{i,j}\}$, $\{b_i, b_j, c_{i,j}\}$ for $\{i, j\} \in \mathcal{P}$, $p = \{a, b\}$, and $\{a_i, b_j, c_{i,j}\}$, $\{b_i, a_j, c_{i,j}\}$ for $\{i, j\} \notin \mathcal{P}$, $p = \{a, b\}$, finally: the sets c_u, c_v, c_w, where $\{u, v, w\}$ is a line of $\mathbf{G}_2(X)$. It is seen that after the identification $s_i \leftrightarrow \{s, i\}$ for $s \in p$, $i \in X$ and $c_{i,j} \leftrightarrow \{i, j\}$ for $i, j \in X$ the structure $\mathbf{M}(X, p, \mathcal{P})$ can be defined on the set $\wp_2(X \cup p)$ and then the configuration is easily seen to be a B_{n+2}-configuration. One can observe that each of the sets

$$S(i) = \{s_i, c_{i,j} \colon j \in X \setminus \{i\},\ s \in p\} \leftrightarrow \{q \in \wp_2(X \cup p) \colon i \in q\}$$

is a complete K_{n+1}-graph freely contained in $\mathbf{M}(X, p, \mathcal{P})$. Moreover, the complement

$$H(i) = \{p, s_j, c_{j,l} \colon s \in p,\ j, l \in X \setminus \{i\} \leftrightarrow \{q \in \wp_2(X \cup p) \colon i \notin q\}$$

of $S(i)$ is a multi-veblen configuration $\mathbf{M}(X \setminus \{i\}, p, \mathcal{P} \restriction (X \setminus \{i\}))$. So, we can write simply $H(i) = \mathcal{H}(\{i\} | X \setminus \{i\})$. It is known that a mutliveblen B_{n+2}-configuration is either a generalized Desargues configuration or it has exactly n maximal freely contained complete graphs.

PROPOSITION 4.5. *Assume that* $\mathbf{M}(X,p,\mathcal{P}) =: \mathfrak{M}$ *is not a generalized Desargues configuration. Then every binomial hyperplane of* $\mathfrak{M}$ *has the form* $\mathcal{H}(\{i\}|X \cup \{p\} \setminus \{i\})$ *with* $i \in X$. *Each hyperplane of* $\mathfrak{M}$ *has form* $\mathcal{H}(A|(X \setminus A) \cup p)$ *for* $\emptyset \neq A \subset X$.

PROOF. It suffices to present $\mathfrak{M}$ in the form $n \bowtie_\xi^\mu \mathbf{G}_2(p)$. Indeed, we observe, first, that $\mathbf{G}_2(p)$ is a trivial structure with a single point and no line. Next, we put $\mu_i(a_i, b_i) = p$ for all $i \in I$, $\{a,b\} = p$. Finally, $\xi(i,j)(a,b) = (a,b)$ when $\{i,j\} \in \mathcal{P}$ and $\xi(i,j)(a,b) = (b,a)$ otherwise. Comparing definitions we see that $\mathbf{M}(X,p,\mathcal{P}) \cong n \bowtie_\xi^\mu \mathbf{G}_2(p)$.

To complete the proof we make use of 3.5 and 3.9: a hyperplane of $\mathfrak{M}$ has form $\mathcal{H}(J|(I \setminus J) \cup p) \pitchfork \mathcal{H}(A|(p \setminus A) \cup I)$ for a subset J of I and an (μ,ξ)-invariant subset A of p. From 3.7 we get that a non void proper subset of p is a one-element set, and such a subset of p is invariant only when $\mathfrak{M}$ is a generalized Desargues configuration, which closes our proof. □

Let us apply 4.5 to the particular case $\mathcal{P} = N_X$ (the empty graph defined on X); it is known that $\mathbf{M}((X,p,\mathcal{P})$ is the structure $\mathbf{V}^*_{|X|}(3)$ dual to the combinatorial Veronesian $\mathbf{V}_3(X)$ (see [14] and Section 4.4). So, $\mathbf{V}^*_3(n)$ has all its binomial hyperplanes of the same geometrical type: the dual Veronesian $\mathbf{V}^*_3(n-1)$.

4.4 Hyperplanes of combinatorial Veronesians

Next, let us pay attention to the class of combinatorial Veronese spaces defined in [14]. Write $X := \{a,b,c\}$ for pairwise distinct a,b,c. Generally, if $f = a^i b^j c^m$ is a multiset with the elements in X we put $|f| = i + j + m$.

The *combinatorial Veronese space* $\mathbf{V}_k(3) = \mathbf{V}_k(\{a,b,c\})$ is the configuration whose points are the multisets $a^i b^j c^m$, $i+j+m = k$: the elements of $\mathfrak{y}_k(\{a,b,c\})$, and whose lines have form eX^i, $i + |e| = k$. It is a $\left(\binom{k+2}{2}_k \binom{k+2}{3}_3\right)$-configuration.

It is known that $\mathbf{V}_2(X) \cong \mathbf{G}_2(4)$ so the hyperplanes of $\mathbf{V}_2(X)$ are, generally, known.

Let $\mathfrak{M} = \mathbf{V}_k(X)$. It is known (cf. [5], [13]) that K_{k+1}-graphs freely contained in $\mathfrak{M}$ have form $\mathfrak{y}_k(A)$, where $A \in \wp_2(X)$, the complement of such a graph is the set $z\mathfrak{y}_{k-1}(X)$, where $\{z\} = X \setminus A$, so it yields a (binomial) subspace of $\mathbf{V}_k(X)$ isomorphic to $\mathbf{V}_{k-1}(X)$.

***Remark* 3.** Note that the set $H = \{a^2c, b^2a, c^2b\}$ yields a hyperplane in every Veblen subconfiguration contained in $\mathbf{V}_3(3)$, but H *is not* a hyperplane of $\mathbf{V}_3(3)$: a 3-element **anticlique** of a 10_3-configuration 'suffices' for at most $3 \times 3 = 9$ lines only (i.e. at most 9 lines intersect such a 3-set).

Let us generate via $\pitchfork$ the hyperplanes, starting form the binomial hyperplanes of a $\mathbf{V}_k(3)$.

- There are three hyperplanes $H_1(u) = u\mathfrak{y}_{k-1}(X)$ and three their complements $\overline{H_1(u)} = \mathfrak{y}_k(\{x,y\})$, $X = \{u,x,y\}$.
- Let us compute: $H_1(x) \pitchfork H_1(y) = \{u^k\} \cup xy\mathfrak{y}_{k-2}(X) =: H_2(u)$, where $X = \{x,y,u\}$.

  The complement of $H_2(u)$ has the form $\overline{H_2(u)} = x\mathfrak{y}_{k-1}(\{u,x\}) \cup y\mathfrak{y}_{k-1}(\{u,y\})$.

- Let us compute: $H_2(a) \pitchfork H_2(b) = H_2(c)$.
- The properties of $\pitchfork$ yield $H_1(x) \pitchfork H_2(y) = H_1(u)$ for $x \neq y$ and $X = \{x, y, u\}$.
- Let us compute again: $H_1(x) \pitchfork H_2(x) = X^k \cup abc\mathfrak{y}_{k-3}(X)$.

We have got seven hyperplanes of $\mathbf{V}_k(X)$.

THEOREM 4.6. *The above are* **all the** *hyperplanes of* $\mathbf{V}_k(X)$. *So,* $\boldsymbol{V}(\mathbf{V}_k(3)) = PG(2,2)$.

PROOF. In the first step we present $\mathfrak{M} := \mathbf{V}_k(X)$ as a system of perspectives of simplices. Recall that $\mathbf{V}_k(X)$ is a B_{k+2}-configuration. In what follows we shall keep a fixed cyclic order $\prec$, say $(a \prec b \prec c \prec a)$ of the elements of X. Note that $\mathbf{G}_2(X) \cong \mathbf{V}_1(X)$ is a single 3-element line. Set $\mathfrak{B} = abc\mathbf{V}_{k-3}(X)$, it is a B_{k-1}-subconfiguration of $\mathbf{V}_k(X)$. Moreover, it is the intersection of three complements of the three maximal complete subgraphs $\mathfrak{y}_k(\{x,y\})$, $\{x,y\} \in \wp_2(X)$ of $\mathfrak{M}$. As usually, we write $\oplus$ for the (partial) binary operation 'the third point on the line through'. Frequently, writing x, yz, z below we mean any x, y, z such that $X = \{x, y, z\}$.

Next, let $Z = \{1, \dots, k-1\}$, then $|Z| = k - 1$. For every $z \in X$ we define

$$\nu_z \colon Z \ni s \longmapsto x^s y^{k-s}, \text{ where } x \prec y,\ \{z, y, z\} = X. \tag{23}$$

So, $\mathfrak{M}$ contains three copies: $F_z = \mathfrak{y}_k(\{x,y\}) \setminus (\mathfrak{y}_k(\{x,z\}) \cup \mathfrak{y}_k(\{y,z\})) = \nu_z(Z)$ $(X = \{x,y,z\})$ of K_Z.

Next, for $z \in Z$ and distinct $i, j \in Z$ we define

$$\mu_z(\{i,j\}) = \mu_z(i,j) = \nu_z(i) \oplus \nu_z(j). \tag{24}$$

It is easy to compute that $\mu_z(i,j) = x^i y^i z^{k-2i} \in abc\mathbf{V}_{k-3}(X)$, so we have defined a surjection $\mu_z \colon \wp_2(Z) \longrightarrow \mathfrak{B}$.

Finally, for distinct $x, y \in X$ and $s \in Z$ we define the map $\xi_{x,y} \colon Z \longrightarrow Z$ by the formula

$$\xi_{x,y}(s) = k - s \tag{25}$$

and we set $\xi_{x,x} = \mathrm{id}$. The following holds for $\{x,y,z\} = X$ and $i, j \in Z$:

$$\nu_x(i), \nu_y(j) \text{ collinear in } \mathfrak{M} \iff j = \xi_{x,y}(i); \quad \nu_x(i) \oplus \nu_y(k-i) = z^k.$$

So, in fact, for each $\{x,y,z\} = X$ we have a perspective $\xi_{x,y} \colon F_x \longrightarrow F_y$ with the centre z^k determined by the formula $\xi_{x,y}(\nu_x(i)) = \nu_y(\xi_{x,y}(i))$. Then $\mathbf{V}_k(X) \cong 3 \bowtie^{\mu}_{\xi} \mathfrak{B}$.

Suppose that $\mathfrak{M}$ contains a hyperplane H of the form $\mathcal{H}(A, X \cup (Z \setminus A))$ with $A \subset Z$. In view of 3.9, A is a (μ, ξ)-invariant subset of Z. Without loss of generality we can assume that $1 \in A$ and then $\{1, k-1\} \subset A$. We get $\mu_z(1, k-1) = xyz^{k-2} = \mu_y(k-1, 2)$; then $2 \in A$, because A is μ-invariant (here, we make use of 3.9(iii'), in fact). Consequently, $k - 2 \in A$ as well.

Step by step, we end up with $\{i, k-i\} \subset A$ for every $i \in Z$, so $A = Z$, which, by 3.5 and 3.9 proves the theorem. $\square$

5 Ideas, hypotheses, and so on ...

5.1 Veldkamp space labeled

As we see, the number of free subgraphs of a BSTS $\mathfrak{M}$ does not determine $\mathfrak{M}$. Also, the number of of its hyperplanes and the types of geometry on hyperplanes do not determine $\mathfrak{M}$. Clearly, $\boldsymbol{V}(\mathfrak{M})$ says only about $|\mathcal{H}(\mathfrak{M})|$.

Recall that if $\mathfrak{M}$ is a B_n-configuration with a free K_{n-1}-subgraph then each hyperplane of $\mathfrak{M}$ is either a B_{n-1} or the union of two unconnected B_{k_1} and B_{k_2}-subconfigurations of $\mathfrak{M}$ with $k_1 + k_2 = n$, $k_1, k_2 \geq 2.$. Suppose that for every $k < n$ we have the list $\mathcal{M}_k$ of B_k-configurations. Let $\mathbf{T}(\mathfrak{M})$ be $\boldsymbol{V}(\mathfrak{M})$ with its points labelled by the types of respective hyperplanes, i.e. by symbols from $\mathcal{M}_{k-1}$ or unordered pairs of symbols from $\mathcal{M}_k \times \mathcal{M}_{n-k}$. It seems that $\mathbf{T}(\mathfrak{M})$ may uniquely characterize $\mathfrak{M}$.

5.2 Problem

In all the examples which were examined in the paper a hyperplane of a BSTS (if exists) is either connected, and then it is a binomial maximal subspace, or it is the union of two unconnected (in a sense: mutually complementary) binomial subspaces. Is this characterization valid for *arbitrary* BSTS.

References

[1] D. BETTEN, U. SCHUMACHER, *The ten configurations* 10_3, Rostock. Math. Kolloq. **46** (1993), 3–10.

[2] F. BUKENHOUT, A. M. COHEN, *Diagram Geometry* (Chapter 10.2). Springer Verlag; 2009.

[3] A. DOLIWA, *The affine Weil group symmetry of Desargues maps and the non-commutative Hirota-Miwa system*, Phys. Lett. A **375** (2011), 1219–1224.

[4] A. DOLIWA, *Desargues maps and the Hirota-Miwa equation*, Proc. R. Soc. A **466** (2010), 1177–1200.

[5] I. GOLONKO, M. PRAŻMOWSKA, *Adjacency in Generalized Projective Veronese Spaces*, Abh. Math. Sem. Univ. Hamb. **76**(2006), 99–114.

[6] H. GROPP, *Configurations and their realization*, *Combinatorics (Rome and Montesilvano, 1994)*, Discrete Math. **174**, no. 1–3 (1997), 137–151.

[7] S. KANTOR, *Die Konfigurationen* $(3, 3)_{10}$, Sitzungsberichte Wiener Akad. **84** (1881), 1291–1314.

[8] K. T. LANMAN, *The taxonomy of Various combinatoric and Geometric Configurations*, Proceedingsof the 2001 Butler University Undergraduate Research Conference. Holcomb Research Institute.

[9] K. PETELCZYC, M. PRAŻMOWSKA, *Twisted Fano spaces and their classification, linear completions of systems of triangle perspectives*, Des. Codes Cryptogr. **54** (2010), 241–251.

[10] 22.K. PETELCZYC, M. PRAŻMOWSKA, K. PRAŻMOWSKI, *A complete classification of the* $(15_4 20_3)$*-configurations with at least three* K_5*-graphs*, Discrete Math. 338 (2015), no. 7, 1243-1251.

[11] K. PETELCZYC, M. PRAŻMOWSKA, 10_3*-configurations and projective realizability of multiplied configurations*, Des. Codes Cryptogr. **51**, no. 1 (2009), 45–54.

[12] B. Polster, *A geometrical picture book*, Springer Verlag, New York, 1998.

[13] M. Prażmowska, K. Prażmowski, *Binomial configurations which contain complete graphs*, arXiv:1404.4064

[14] M. Prażmowska, K. Prażmowski, *Combinatorial Veronese structures, their geometry, and problems of embeddability*, Results Math. **51** (2008), 275–308.

[15] M. Prażmowska, K. Prażmowski, *Some generalization of Desargues and Veronese configurations*, Serdica Math. J. **32** (2006), no 2–3, 185–208.

[16] M. Prażmowska, K. Prażmowski, *Semi-Pappus configurations; combinatorial generalizations of the Pappus configuration*, Des. Codes Cryptogr. 61 (2011), no. 1, 91-103.

[17] M. Prażmowska, *Multiple perspectives and generalizations of the Desargues configuration*, Demonstratio Math. **39** (2006), no. 4, 887–906.

[18] M. Prażmowska, *On the existence of projective embeddings of some multiveblen configurations*, Bull. Belg. Math. Soc. Simon-Stevin **17** (2010), 1–15.

[19] M. Prażmowska, *On some regular multi-veblen configurations, the geometry of combinatorial quasi Grassmannians*, Demonstratio Math. **42** (2009), no.2 387–402.

[20] M. Ronan, *Embeddings and hyperplanes of discrete geometries*, European J. Combin. 8 (1987), 179–185.

[21] M. Saniga, F. Holweck, P. Pracna, *Cayley-Dickson Algebras and Finite Geometry*, arXiv:1405.6888.

[22] E. Shult, *On Veldkamp Lines*, Bull. Belg. Math. Soc. Simon Stevin 4 (1997), 299–316.

[23] E. Steinitz, *Konfigurationen der projektiven Geometrie*, Encyclop. d. math. Wiss. III (Geometrie) 1910, 481–516.

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